

SO(3,2)/Sp(2) symmetries in BTZ black holes

Soon-Tae Hong*

Department of Science Education, Ewha Womans University, Seoul 120-750 Korea

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We study global flat embeddings, three accelerations and Hawking temperatures of the BTZ black holes in the framework of two-time physics scheme associated with Sp(2) local symmetry, to construct their corresponding SO(3,2) global symmetry invariant Lagrangians both inside and outside event horizons. Moreover, the Sp(2) local symmetry is discussed in terms of the metric time-independence.

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I. INTRODUCTION

There have been tremendous progresses in lower dimensional black holes associated with the string theory since an exact conformal field theory describing a black hole in two-dimensional space-time was proposed [1]. Especially, the (2+1) dimensional Banados-Teitelboim-Zanelli (BTZ) black holes [2, 3, 4] have enjoyed lots of successes in relativity and string communities, since thermodynamics of higher dimensional black holes can be interpreted in terms of the BTZ black hole solutions. In fact, the dual solutions of the BTZ black holes are related to the solutions in the string theory, so-called (2+1) black strings [5, 6].

On the other hand, the higher dimensional global flat embeddings of the black hole solutions are subjects of great interest both to mathematicians and to physicists. In differential geometry, it has been well-known that four dimensional Schwarzschild metric is not embedded in R^5 [7]. Recently, (5+1) dimensional global embedding Minkowski space structure for the Schwarzschild black hole has been obtained [8] to investigate a thermal Hawking effect on a curved manifold [9] associated with an Unruh effect [10] in these higher dimensional space time. It has been also shown that the uncharged and charged BTZ black holes are embedded in (2+2) [8] and (3+2) dimensions,¹ while the uncharged and charged black strings are embedded in (3+1) and (3+2) dimensions [12], respectively. Note that one can have two time coordinates in these embedding solutions to suggest so-called two-time physics [13]. Historically, the two-time physics was formulated long ago when the (3+1) Maxwell theory on a conformally invariant (4+2) manifold was constructed [14]. Recently, the two-time physics scheme has been applied to M theory [15] and noncommutative gauge theories [16].

In this paper we will investigate symmetries involved in the BTZ black hole embeddings such as SO(3,2) global and Sp(2) local symmetries. In section 2, we will study complete embedding solutions, three accelerations and Hawking temperatures “inside and outside” the event horizons in the framework of the two-time physics and then construct the SO(3,2) global symmetry invariant Lagrangians associated with these embedding solutions in section 3. The Sp(2) local symmetry will also be discussed in terms of the metric time-independence. In Appendix, we will revisit the charged BTZ black hole to construct its minimal embedding solution.

II. COMPLETE FLAT EMBEDDING GEOMETRIES

We first briefly recapitulate the global flat embedding solution given in [8], for the (2+1) rotating BTZ black hole [2, 3] which is described by 3-metric

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \quad (2.1)$$

where the lapse and shift functions are

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2}, \quad (2.2)$$

*Electronic address: soonhong@ewha.ac.kr

¹ This (3+2) minimal embedding is constructed in Appendix. In the previous work [11], the charged BTZ black hole has (3+3) nonminimal embedding which is, however, reduced to the uncharged BTZ embedding in the vanishing charge limit.

respectively. Note that for the nonextremal case there exist two horizons $r_{\pm}(J)$ satisfying the following equations,

$$0 = -M + \frac{r_{\pm}^2}{l^2} + \frac{J^2}{4r_{\pm}^2}, \quad (2.3)$$

in terms of which we can rewrite the lapse and shift functions as follows

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 l^2}, \quad N^\phi = -\frac{r_+ r_-}{r^2 l}. \quad (2.4)$$

Here one notes that this BTZ space originates from Anti-de Sitter one via the geodesic identification $\phi = \phi + 2\pi$. The (2+2) minimal BTZ global flat embedding $ds^2 = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 - (dz^3)^2$ is then given by the coordinate transformations for $r \geq r_+$ as follows

$$\begin{aligned} z^0 &= l \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ z^1 &= l \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ z^2 &= l \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\ z^3 &= l \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right). \end{aligned} \quad (2.5)$$

In the following we will construct complete embedding solutions, three accelerations and Hawking temperatures “inside and outside” the event horizons.

A. Case I: $r \geq r_+$

In the two-time physics scheme [13], we consider global flat embedding structure for the (2+1) rotating BTZ black hole whose metric is now given by

$$ds^2 = \eta_{MN} dX^M dX^N \quad (2.6)$$

$$X^M = l(c_- \cosh R, s_- \cosh R, s_+ \sinh R, c_+ \sinh R, 1) \quad (2.7)$$

where $\eta_{MN} = \text{diag}(-1, +1, -1, +1, +1)$ with $M = (0', 1', 0, 1, 2)$ and

$$c_{\pm} = \cosh \frac{x^{\pm}}{l}, \quad s_{\pm} = \sinh \frac{x^{\pm}}{l}. \quad (2.8)$$

With the canonical momenta p_{μ} conjugate to x_{μ} given as

$$x^{\mu} = \left(\frac{x^+}{l}, R, \frac{x^-}{l} \right), \quad p_{\mu} = (p_+, p_R, p_-), \quad (2.9)$$

we can construct

$$\begin{aligned} P^M &= \frac{1}{l} (c_- p_R \sinh R + s_- p_- \text{sech} R, s_- p_R \sinh R + c_- p_- \text{sech} R, \\ &\quad s_+ p_R \cosh R - c_+ p_+ \text{cosech} R, c_+ p_R \cosh R - s_+ p_+ \text{cosech} R, 0). \end{aligned} \quad (2.10)$$

to satisfy the $\text{Sp}(2)$ local symmetry associated with the two-time physics,²

$$X_M X^M = 0, \quad X_M P^M = 0, \quad P_M P^M = 0. \quad (2.11)$$

² Here one can easily show that $X_M X^M = X_M P^M = 0$. The last symmetry condition $P_M P^M = 0$ will be discussed later.

Now we differentiate X^M in (2.7) to yield

$$ds^2 = dX_M dX^M = l^2 dR^2 - \sinh^2 R (dx^+)^2 + \cosh^2 R (dx^-)^2, \quad (2.12)$$

which, using the identification for x^\pm

$$x^\pm = \frac{r_\pm}{l} t - r_\mp \phi, \quad (2.13)$$

can be rewritten as for $r \geq r_+$

$$\begin{aligned} ds^2 &= l^2 dR^2 + l^{-2} (r_-^2 \cosh^2 R - r_+^2 \sinh^2 R) dt^2 + 2r^2 N^\phi dt d\phi \\ &\quad + (r_+^2 \cosh^2 R - r_-^2 \sinh^2 R) d\phi^2. \end{aligned} \quad (2.14)$$

Exploiting the ansatz³ for $\cosh R$ and $\sinh R$

$$\cosh R = \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2}, \quad \sinh R = \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \quad (2.15)$$

we can reproduce the BTZ metric (2.1) and the (3+2) global flat embedding for $r \geq r_+$

$$\begin{aligned} X^{0'} &= l \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\ X^{1'} &= l \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\ X^0 &= l \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ X^1 &= l \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ X^2 &= l, \end{aligned} \quad (2.16)$$

to yield the identification between the standard global flat embedding (2.5) and that of two-time physics: $(X^{0'}, X^{1'}, X^0, X^1) = (z^3, z^2, z^0, z^1)$. Here one notes that $X^2 = l$ does not contribute the minimal global flat embedding since it is constant, and this coordinate X^2 serves to fulfill the $\text{Sp}(2)$ symmetry (2.11).

Next, introducing the Killing vector $\xi = \partial_t - N^\phi \partial_\phi$ we evaluate the three acceleration

$$a_3 = \frac{r^4 - r_+^2 r_-^2}{r^2 l (r^2 - r_+^2)^{1/2} (r^2 - r_-^2)^{1/2}}, \quad (2.17)$$

and the Hawking temperature [9]

$$T_H = \frac{a_4}{2\pi} = \frac{r(r_+^2 - r_-^2)}{2\pi r_+ l (r^2 - r_+^2)^{1/2} (r^2 - r_-^2)^{1/2}}, \quad (2.18)$$

which are consistent with the fact that the a_4 is also attainable from the relation [9]

$$a_4 = \frac{k}{g_{00}^{1/2}}. \quad (2.19)$$

³ In the literature [17], there appears a brief sketch on the BTZ embedding outside the horizon in the two-time physics scheme, without explicit construction of $\cosh R$ and $\sinh R$ in (2.15) for instance.

B. Case II: $r_- \leq r \leq r_+$

Next, we consider the global flat embedding of the BTZ black hole in the range of $r_- \leq r \leq r_+$ by exploiting a little bit different choice for X^M

$$X^M = l(c_- \cos R, s_- \cos R, c_+ \sin R, s_+ \sin R, 1) \quad (2.20)$$

which satisfies the $\text{Sp}(2)$ local symmetry (2.11). As in the previous section, differentiating X^M in (2.20) yields for $r_- \leq r \leq r_+$

$$ds^2 = -l^2 dR^2 + l^{-2}(r_+^2 \sin^2 R + r_-^2 \cos^2 R) dt^2 + 2r^2 N^\phi dt d\phi + (r_+^2 \cos^2 R + r_-^2 \sin^2 R) d\phi^2. \quad (2.21)$$

Exploiting the ansatz for $\cos R$ and $\sin R$

$$\cos R = \left(\frac{r_-^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2}, \quad \sin R = \left(\frac{r_+^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \quad (2.22)$$

we can obtain the (3+2) global flat embedding for $r_- \leq r \leq r_+$

$$\begin{aligned} X^{0'} &= l \left(\frac{r_-^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\ X^{1'} &= l \left(\frac{r_-^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\ X^0 &= l \left(\frac{r_+^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ X^1 &= l \left(\frac{r_+^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\ X^2 &= l. \end{aligned} \quad (2.23)$$

Next, similar to the $r \geq r_+$ case, with the Killing vector $\xi = \partial_t - N^\phi \partial_\phi$ we evaluate the three acceleration

$$a_3 = \frac{r^4 - r_+^2 r_-^2}{r^2 l (r_+^2 - r^2)^{1/2} (r^2 - r_-^2)^{1/2}}, \quad (2.24)$$

and the Hawking temperature

$$T_H = \frac{a_4}{2\pi} = \frac{r(r_+^2 - r_-^2)}{2\pi r_+ l (r_+^2 - r^2)^{1/2} (r^2 - r_-^2)^{1/2}}. \quad (2.25)$$

C. Case III: $r \leq r_-$

Similarly, for the case of the range inside the inner horizon, $r \leq r_-$, we introduce

$$X^M = l(s_- \sinh R, c_- \sinh R, c_+ \cosh R, s_+ \cosh R, 1) \quad (2.26)$$

to yield

$$ds^2 = l^2 dR^2 + l^{-2}(r_+^2 \cosh^2 R - r_-^2 \sinh^2 R) dt^2 + 2r^2 N^\phi dt d\phi + (r_-^2 \cosh^2 R - r_+^2 \sinh^2 R) d\phi^2. \quad (2.27)$$

With the ansatz for $\cosh R$ and $\sinh R$

$$\cosh R = \left(\frac{r_+^2 - r_-^2}{r_+^2 - r_-^2} \right)^{1/2}, \quad \sinh R = \left(\frac{r_-^2 - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \quad (2.28)$$

we can construct the (3+2) global flat embedding for $r \leq r_-$

$$\begin{aligned}
X^{0'} &= l \left(\frac{r_-^2 - r^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\
X^{1'} &= l \left(\frac{r_-^2 - r^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_-}{l^2} t - \frac{r_+}{l} \phi \right), \\
X^0 &= l \left(\frac{r_+^2 - r^2}{r_+^2 - r_-^2} \right)^{1/2} \cosh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\
X^1 &= l \left(\frac{r_+^2 - r^2}{r_+^2 - r_-^2} \right)^{1/2} \sinh \left(\frac{r_+}{l^2} t - \frac{r_-}{l} \phi \right), \\
X^2 &= l.
\end{aligned} \tag{2.29}$$

Next, similar to the above cases, with the Killing vector $\xi = \partial_t - N^\phi \partial_\phi$ we evaluate the three acceleration

$$a_3 = \frac{r_+^2 r_-^2 - r^4}{r^2 l (r_+^2 - r^2)^{1/2} (r_-^2 - r^2)^{1/2}}, \tag{2.30}$$

and the Hawking temperature

$$T_H = \frac{a_4}{2\pi} = \frac{r(r_+^2 - r_-^2)}{2\pi r_+ l (r_+^2 - r^2)^{1/2} (r_-^2 - r^2)^{1/2}}. \tag{2.31}$$

This completes the full global flat embeddings of the BTZ black hole in the two-time physics scheme. Note that the above (3+2) BTZ embedding solutions are consistent with those in [3] where they obtained the (2+2) BTZ embeddings in the standard global flat embedding scheme, without considering the $\text{Sp}(2)$ local symmetry (2.11) and the corresponding $\text{SO}(3,2)$ global symmetry invariant Lagrangian construction, which will be discussed in the two-time physics approach [13] in the next section.

III. $\text{SO}(3,2)/\text{SP}(2)$ SYMMETRIES

A. Case I: $r \geq r_+$

Now, in the global flat embedding (2.16) for $r \geq r_+$, we consider the Lorentz generators of the $\text{SO}(3,2)$ symmetry

$$L^{MN} = X^M P^N - X^N P^M, \tag{3.1}$$

which, using the conjugate pair (X^M, P^M) in (2.7) and (2.10), yields

$$\begin{aligned}
L^{2M} &= l P^M, \quad L^{0'1'} = p_-, \quad L^{01} = p_+, \\
L^{0'0} &= c_- s_+ p_R - c_- c_+ p_+ \coth R - s_- s_+ p_- \tanh R, \\
L^{1'1} &= s_- c_+ p_R - s_- s_+ p_+ \coth R - c_- c_+ p_- \tanh R, \\
L^{0'1} &= c_- c_+ p_R - c_- s_+ p_+ \coth R - s_- c_+ p_- \tanh R, \\
L^{1'0} &= s_- s_+ p_R - s_- c_+ p_+ \coth R - c_- s_+ p_- \tanh R,
\end{aligned} \tag{3.2}$$

where we have used $\cosh R$ and $\sinh R$ in (2.15). Note that these Lorentz generators produce the classical $\text{SO}(3,2)$ global symmetry transformations under δ defined as the Poisson bracket $\delta = \frac{1}{2} \epsilon_{MN} \{L^{MN}, \}$ as follows⁴

$$\delta \left(\frac{x^+}{l} \right) = (\epsilon_{0'0} c_- c_+ + \epsilon_{0'1} c_- s_+ + \epsilon_{1'0} s_- c_+ + \epsilon_{1'1} s_- s_+) \coth R$$

⁴ In the literature [3], the Lorentz generators of the $\text{SO}(2,2)$ subgroup are constructed in the standard global flat embedding scheme.

$$\begin{aligned}
& -\epsilon_{01} - (\epsilon_{02}c_+ + \epsilon_{12}s_+)\text{cosech}R, \\
\delta R &= -\epsilon_{0'0}c_-s_+ - \epsilon_{0'1}c_-c_+ - \epsilon_{1'0}s_-s_+ - \epsilon_{1'1}s_-c_+ \\
& + (\epsilon_{0'2}c_- + \epsilon_{1'2}s_-)\sinh R + (\epsilon_{02}s_+ + \epsilon_{12}c_+)\cosh R, \\
\delta\left(\frac{x^-}{l}\right) &= (\epsilon_{0'0}s_-s_+ + \epsilon_{0'1}s_-c_+ + \epsilon_{1'0}c_-s_+ + \epsilon_{1'1}c_-c_+)\tanh R \\
& - \epsilon_{0'1'} + (\epsilon_{0'2}s_- + \epsilon_{1'2}c_-)\text{sech}R.
\end{aligned} \tag{3.3}$$

On the other hand, in the two-time physics the SO(3,2) Lagrangian is given by

$$L = \dot{X}^M P_M - \frac{1}{2}A_{22}P^M P_M, \tag{3.4}$$

which, exploiting the conjugate pair (X^M, P^M) in (2.7) and (2.10), yields

$$L = \frac{\dot{x}^+}{l}p_+ + \dot{R}p_R + \frac{\dot{x}^-}{l}p_- - \frac{1}{2}A_{22}g^{\mu\nu}p_\mu p_\nu. \tag{3.5}$$

Here the conjugate momenta p_μ is defined in (2.9) and the metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \frac{1}{l^2}\text{diag}(-\text{cosech}^2 R, 1, \text{sech}^2 R). \tag{3.6}$$

The above Lagrangian (3.5) can also be rewritten as

$$L = \frac{1}{2A_{22}}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{3.7}$$

with x^μ defined in (2.9) and the inverse metric $g_{\mu\nu}$.

After some algebra, we obtain the transformation rule for the A_{22} as below

$$\delta A_{22} = 2A_{22}[(\epsilon_{0'2}c_- + \epsilon_{1'2}s_-)\cosh R + (\epsilon_{02}s_+ + \epsilon_{12}c_+)\sinh R], \tag{3.8}$$

from which, together with the above transformation rules (3.3), one can easily see that the Lagrangian (3.4) is SO(3,2) global symmetry invariant.

B. Case II: $r_- \leq r \leq r_+$

Next, we consider the global flat embedding (2.23) for $r_- \leq r \leq r_+$. As in the previous case, constructing P^M conjugate to X^M in (2.20) as below

$$\begin{aligned}
P^M &= \frac{1}{l}(c_-p_R \sin R + s_-p_- \sec R, s_-p_R \sin R + c_-p_- \sec R, \\
& -c_+p_R \cos R + s_+p_+ \csc R, -s_+p_R \cos R + c_+p_+ \csc R, 0),
\end{aligned} \tag{3.9}$$

which satisfies the Sp(2) local symmetry (2.11), we can obtain the Lorentz generators of the SO(3,2) global symmetry

$$\begin{aligned}
L^{2M} &= lP^M, \quad L^{0'1'} = p_-, \quad L^{01} = p_+, \\
L^{0'0} &= -c_-c_+p_R + c_-s_+p_+\cot R - s_-c_+p_- \tan R, \\
L^{1'1} &= -s_-s_+p_R + s_-c_+p_+\cot R - c_-s_+p_- \tan R, \\
L^{0'1} &= -c_-s_+p_R + c_-c_+p_+\cot R - s_-s_+p_- \tan R, \\
L^{1'0} &= -s_-c_+p_R + s_-s_+p_+\cot R - c_-c_+p_- \tan R,
\end{aligned} \tag{3.10}$$

to yield the classical SO(3,2) global symmetry transformations for x^μ in (2.9)

$$\delta\left(\frac{x^+}{l}\right) = -(\epsilon_{0'0}c_-s_+ + \epsilon_{0'1}c_-c_+ + \epsilon_{1'0}s_-s_+ + \epsilon_{1'1}s_-c_+)\cot R$$

$$\begin{aligned}
& -\epsilon_{01} + (\epsilon_{02}s_+ + \epsilon_{12}c_+)\operatorname{cosec}R, \\
\delta R &= \epsilon_{0'0}c_-c_+ + \epsilon_{0'1}c_-s_+ + \epsilon_{1'0}s_-c_+ + \epsilon_{1'1}s_-s_+ \\
& + (\epsilon_{0'2}c_- + \epsilon_{1'2}s_-)\sin R - (\epsilon_{02}c_+ + \epsilon_{12}s_+)\cos R, \\
\delta\left(\frac{x^-}{l}\right) &= (\epsilon_{0'0}s_-c_+ + \epsilon_{0'1}s_-s_+ + \epsilon_{1'0}c_-c_+ + \epsilon_{1'1}c_-s_+)\tan R \\
& - \epsilon_{0'1'} + (\epsilon_{0'2}s_- + \epsilon_{1'2}c_-)\sec R.
\end{aligned} \tag{3.11}$$

On the other hand, inserting the conjugate pair (X^M, P^M) in (2.20) and (3.9) into the Lagrangian (3.4), we obtain the metric

$$g^{\mu\nu} = \frac{1}{l^2} \operatorname{diag} (\operatorname{cosec}^2 R, -1, \sec^2 R), \tag{3.12}$$

from which we also obtain the $\text{SO}(3,2)$ global symmetry transformations for A_{22}

$$\delta A_{22} = 2A_{22}[(\epsilon_{0'2}c_- + \epsilon_{1'2}s_-)\cos R + (\epsilon_{02}c_+ + \epsilon_{12}s_+)\sin R]. \tag{3.13}$$

Exploiting the transformation rules (3.11) and (3.13), one can easily see that the Lagrangian (3.4) with X^M and P^M in (2.20) and (3.9) is invariant under the $\text{SO}(3,2)$ global symmetry transformations.

C. Case III: $r \leq r_-$

Finally, we consider the (3+2) global flat embedding for $r \leq r_-$. Exploiting X^M in (2.29) and constructing its conjugate momenta P^M as below

$$\begin{aligned}
P^M &= \frac{1}{l}(s_-p_R \cosh R - c_-p_- \operatorname{cosech}R, c_-p_R \cosh R - s_-p_- \operatorname{cosech}R, \\
& c_+p_R \sinh R + s_+p_+ \operatorname{sech}R, s_+p_R \sinh R + c_+p_+ \operatorname{sech}R, 0),
\end{aligned} \tag{3.14}$$

to satisfy the $\text{Sp}(2)$ symmetry (2.11), we can obtain the Lorentz generators of the $\text{SO}(3,2)$ symmetry

$$\begin{aligned}
L^{2M} &= lP^M, \quad L^{0'1'} = p_-, \quad L^{01} = p_+, \\
L^{0'0} &= -s_-c_+p_R + s_-s_+p_+ \tanh R + c_-c_+p_- \coth R, \\
L^{1'1} &= -c_-s_+p_R + c_-c_+p_+ \tanh R + s_-s_+p_- \coth R, \\
L^{0'1} &= -s_-s_+p_R + s_-c_+p_+ \tanh R + c_-s_+p_- \coth R, \\
L^{1'0} &= -c_-c_+p_R + c_-s_+p_+ \tanh R + s_-c_+p_- \coth R.
\end{aligned} \tag{3.15}$$

Inserting the Lorentz generators (3.15) into the classical $\text{SO}(3,2)$ global symmetry transformation rules $\delta = \frac{1}{2}\epsilon_{MN}\{L^{MN}, \}$ yields

$$\begin{aligned}
\delta\left(\frac{x^+}{l}\right) &= -(\epsilon_{0'0}s_-s_+ + \epsilon_{0'1}s_-c_+ + \epsilon_{1'0}c_-s_+ + \epsilon_{1'1}c_-c_+)\tanh R \\
& - \epsilon_{01} + (\epsilon_{02}s_+ + \epsilon_{12}c_+)\operatorname{sech}R, \\
\delta R &= \epsilon_{0'0}s_-c_+ + \epsilon_{0'1}s_-s_+ + \epsilon_{1'0}c_-c_+ + \epsilon_{1'1}c_-s_+ \\
& + (\epsilon_{0'2}s_- + \epsilon_{1'2}c_-)\cosh R + (\epsilon_{02}c_+ + \epsilon_{12}s_+)\sinh R, \\
\delta\left(\frac{x^-}{l}\right) &= -(\epsilon_{0'0}c_-c_+ + \epsilon_{0'1}c_-s_+ + \epsilon_{1'0}s_-c_+ + \epsilon_{1'1}s_-s_+)\coth R \\
& - \epsilon_{0'1'} - (\epsilon_{0'2}c_- + \epsilon_{1'2}s_-)\operatorname{cosech}R.
\end{aligned} \tag{3.16}$$

On the other hand, substituting the conjugate pair (X^M, P^M) in (2.26) and (3.14) into the Lagrangian (3.4), we obtain the metric

$$g^{\mu\nu} = \frac{1}{l^2} \operatorname{diag} (\operatorname{sech}^2 R, 1, -\operatorname{cosech}^2 R), \tag{3.17}$$

so that we can construct the $\text{SO}(3,2)$ global symmetry transformations for A_{22}

$$\delta A_{22} = 2A_{22}[(\epsilon_{0'2}s_- + \epsilon_{1'2}c_-)\sinh R + (\epsilon_{02}c_+ + \epsilon_{12}s_+)\cosh R]. \tag{3.18}$$

In the region inside the inner horizon where X^M and P^M are given by (2.29) and (3.14), as in the previous cases, we can thus obtain the $\text{SO}(3,2)$ global symmetry invariant Lagrangian under the transformation rules (3.16) and (3.18).

Now it seems appropriate to comment on the $\text{Sp}(2)$ local symmetry associated with the two-time physics, for the BTZ global embedding solutions. To be more specific, we consider X^M and P^M for $r \geq r_+$ in (2.7) and (2.10) to evaluate

$$P_M P^M = l^{-2}(-p_+^2 \text{cosech}^2 R + p_R^2 + p_-^2 \text{sech}^2 R). \quad (3.19)$$

Exploiting the metric $g^{\mu\nu}$ in (3.6), we can rewrite p_μ in (2.9) in terms of the \dot{x}^μ

$$(p^+, p_R, p^-) = \frac{l^2}{A_{22}} \left(-\frac{\dot{x}^+}{l} \sinh^2 R, \dot{R}, \frac{\dot{x}^-}{l} \cosh^2 R \right), \quad (3.20)$$

to, together with the explicit expressions for $\cosh R$ and $\sinh R$ in (2.15), yield a relation between the $P_M P^M$ and the BTZ metric

$$P_M P^M = \frac{1}{A_{22}^2} \left(\frac{ds}{dt} \right)^2, \quad (3.21)$$

which vanishes since the BTZ metric or line element itself is time-independent. We can thus explicitly show that the $\text{Sp}(2)$ symmetry (2.11) is conserved in the (3+2) BTZ global embedding solutions. Moreover, the Lagrangian (3.4) can be rewritten as

$$L = \frac{1}{2} A_{22} P_M P^M. \quad (3.22)$$

Here one notes that, due to the BTZ metric time-independence, the $\text{SO}(3,2)$ Lagrangian or Hamiltonian vanishes, which is a characteristic of the two-time physics [13].

IV. CONCLUSIONS

In conclusion, in the framework of two-time physics scheme, we have explicitly obtained the global flat embeddings, three accelerations and Hawking temperatures of the BTZ black holes both inside and outside the event horizons by exploiting the $\text{Sp}(2)$ local symmetry. Moreover, we have constructed the $\text{SO}(3,2)$ global symmetry invariant Lagrangians associated with these BTZ black hole embedding solutions. The $\text{Sp}(2)$ local symmetry has been also discussed in terms of the metric time-independence.

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APPENDIX A

Now we consider the charged BTZ black hole with 3-metric

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2, \quad (A.1)$$

where the charged lapse function is given as [2, 3, 4]

$$N^2 = -M + \frac{r^2}{l^2} - 2q^2 \ln r, \quad (A.2)$$

where the mass M can be rewritten as $M = r_H^2/l^2 - 2q^2 \ln r_H$ with the horizon $r_H(q)$, which is the root of $-M + r^2/l^2 - 2q^2 \ln r = 0$. The surface gravity in this charged BTZ black hole is given by $k_H = [(r_H/l)^2 - q^2]/r_H$. Making an ansatz of four coordinates (z^0, z^1, z^2, z^3) in (A.3), which is different from the previous one in [11], we can construct

the (3+2) global flat embedding $ds^2 = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 - (dz^4)^2$ with the following coordinate transformations for $ql \leq r_H < r$

$$\begin{aligned} z^0 &= k_H^{-1} \left(-M + \frac{r^2}{l^2} - 2q^2 \ln r \right)^{1/2} \sinh k_H t, \\ z^1 &= k_H^{-1} \left(-M + \frac{r^2}{l^2} - 2q^2 \ln r \right)^{1/2} \cosh k_H t, \\ z^2 &= r \cos \phi, \\ z^3 &= r \sin \phi, \\ z^4 &= \int dr \left[\frac{(1 + q^2 l^2 / r r_H)(1 - q^2 l^2 / r r_H)}{k_H^2 l^2 [1 - (q^2 l^2 / r_H^2) n(r, r_H)]} + 1 \right]^{1/2}. \end{aligned} \quad (\text{A.3})$$

Here $n(r, r_H)$ is given by

$$n(r, r_H) = \frac{2r_H^2}{r^2 - r_H^2} \ln \frac{r}{r_H}, \quad (\text{A.4})$$

which, due to L'Hospital's rule, approaches unity as r goes to r_H .

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